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Estimation of the Thermal Diffusivity Profile in Functionally Gradient Materials with the Stepwise Heating Method

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Temperature response in a functionally gradient material (FGM) which is subjected to stepwise heating is investigated, to estimate the profile of the thermal diffusivity from the temperature response at the rear surface of the FGM. Emphasis is placed on a distribution parameter which gives the profiles of the thermophysical properties when an exact analytical solution exists for the temperature response in the FGM. An explicit expression to determine the distribution parameter is obtained as a function of the thermophysical properties at the rear surface. This explicit expression can represent the dependence of the temperature response on the thermophysical properties within 5% in comparison to the exact solution. It is expected that this identification can provide useful insight into the estimation of thermophysical properties in FGMs. The usefulness of this relation is also examined by comparing given and estimated profiles for the thermal diffusivity. Fair agreement is demonstrated as far as the trend and the approximate magnitude are concerned.

KEY WORDS: inverse problem; functionally gradient material (FGM); stepwise heating method; thermal diffusivity.

1. INTRODUCTION

Functionally gradient materials (FGM), which are composed of different material components such as ceramics and metals with continuous profiles in composition, have attracted special interest as advanced heat-shielding/structural materials in future space applications, electronic materials, and others. For an FGM to qualify as an advanced heat-shielding/structural

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material, the thermophysical properties as well as the mechanical properties should be evaluated properly. Transient methods, such as those involving pulse- or step-wise heating, can be used for these evaluations because of their simplicity and advantages at high temperatures. However, application of the transient methods requires great care because the thermal diffusivity obtained from the temperature response is different from that of the averaged property related to the thermal resistance. Therefore, before making measurements, it is essential to investigate and confirm the measurement principles in the application of the transient methods to FGM.

For this purpose, a series of investigations has been conducted by the present authors. First, dominant parameters in the transient methods were identified by investigating the temperature response in the two-layered material [1]. Next, prior to the investigation of the FGM, a general analytical solution for the temperature response in the multilayered material which is subjected to transient heating was derived [2]. This identification provided useful insight into the dependence of the temperature response on the thermophysical properties of each layer and facilitated not only confirmation of measurement principles but also future analytical studies to evaluate various effects in the measurements. Then the transient temperature response in the FGM was investigated with the approximate solution [2] derived by considering an infinitesimal thickness for each layer. From the engineering point of view, the validity and/or applicability of this solution was further examined by comparing it with the exact analytical solution for FGM in which thermophysical properties have certain profiles. These were examined by Ishiguro et al. [3] for the pulse-wise heating method and by Araki et al. [4] for the stepwise heating method.

The present study can be considered to be an extension of our previous work on the FGM which has an exact analytical solution for the temperature response. The objective is to estimate a profile for a thermophysical property by evaluating its distribution parameter, using the temperature response at the rear surface of the FGM, which is subjected to stepwise heating at the front surface. For simplicity, we consider the situation in which the thermal diffusivity varies along the distance from the rear surface, while the heat capacity is constant. An explicit expression to determine the distribution parameter is obtained as a function of the thermophysical properties at the rear surface. To demonstrate further the usefulness of this expression, the profile of the thermal diffusivity is estimated from the temperature response which is numerically obtained for another FGM, whose profiles of thermophysical properties are given in another way. Reasonable agreement is demonstrated between the given and the estimated profiles for thermal diffusivity.

2. TEMPERATURE RESPONSE IN FGM

It was shown in Ref. 3 that there exists an exact analytical solution for the temperature response in the FGM when the profile of the heat-penetration coefficient Λ $(=\lambda/\sqrt{a})$ is expressed as

$$(\Lambda/\Lambda_R)=\exp(2\alpha\zeta) \tag{1}$$

where

$$\zeta=\frac{1}{\eta_L}\int_z^0\frac{dz}{\sqrt{a(z)}};\qquad \eta_L=\int_{-L}^0\frac{dz}{\sqrt{a(z)}} \tag{2}$$

Here α is the distribution parameter, ζ the normalized thermal diffusion time form a certain point in the FGM to its rear surface, η_L is the total thermal diffusion time, λ the thermal conductivity, a the thermal diffusivity, L the thickness of the FGM, and z the distance from the rear surface, and the subscripts F and R designate the front and rear surfaces, respectively. Figure 1 shows profiles of (Λ/Λ_R) as a function of ζ, with α taken as a parameter.

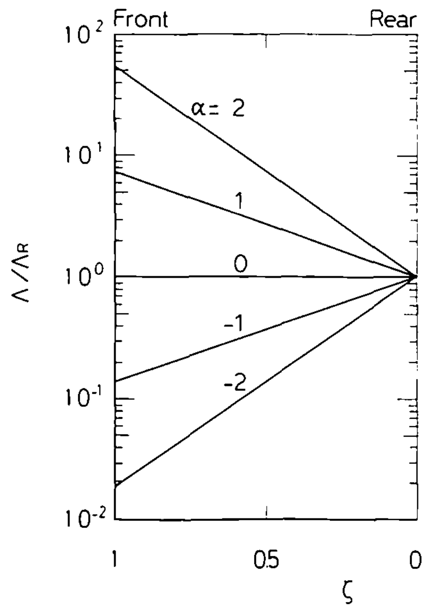

Fig. 1. Profile of the heat-penetration coefficient Λ in the FGM as a function of the normalized thermal diffusion time ζ, with α taken as a parameter.

For stepwise heating, together with the positive root of $k\pi$ for the characteristic equation, the temperature response is expressed as

$$\theta(z, t) = \frac{\eta_L Q \alpha \exp(\alpha)}{A_F \sinh \alpha} V_S \tag{3}$$

$$V_S = \mathrm{Fo} + \frac{1 + \exp(-2\alpha\zeta) + 2\alpha(\zeta - \coth \alpha)}{4\alpha^2} - 2 \sum_{k=1}^{\infty} \frac{(-1)^k \cos(k\pi\zeta)}{(k\pi)^2} \exp[-(k\pi)^2 \mathrm{Fo}] f_k \tag{4}$$

$$f_k = \left(\frac{\sinh \alpha}{\alpha}\right) \exp(-\alpha\zeta - \alpha^2 \mathrm{Fo}) \left[\frac{(k\pi)^2}{(k\pi)^2 + \alpha^2}\right]^2 \left[1 + \frac{\alpha}{k\pi} \tan(k\pi\zeta)\right] \tag{5}$$

where Fo is the Fourier number defined as

$$\mathrm{Fo} = \frac{t}{\eta_L^2} \tag{6}$$

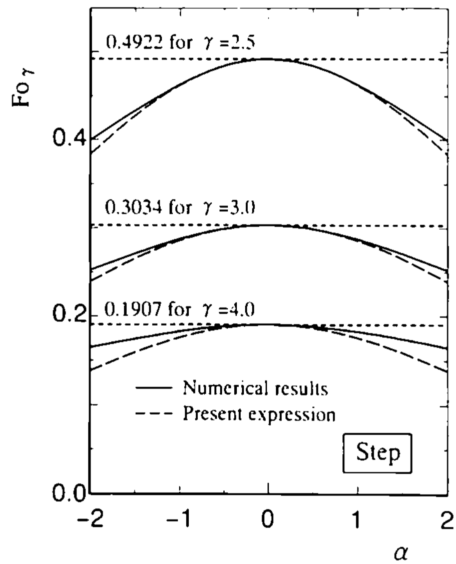

Fig. 2. The Fourier number Fo_γ as a function of α, with the temperature ratio at the rear surface taken as a parameter. The solid curve was obtained by numerical calculations; the dashed curve, by the approximate expression, Eq. (15).

The Fourier number for a certain temperature ratio γ [$= V_S(2\text{Fo})/V_S(\text{Fo})$] at the rear surface has already been obtained as a function of α, as shown in Fig. 2 [4] by solid curves.

3. ESTIMATION OF THE DISTRIBUTION PARAMETER

In this section, we estimate the distribution parameter α from the temperature response at the rear surface of the FGM, restricting ourselves to the stepwise heating method. For simplicity, we consider the situation in which the thermal diffusivity varies along the distance from the rear surface of the FGM, while the volume heat capacity (ρc) is constant. Under this assumption, we have the following set of relations:

$$\sqrt{a_F} - \sqrt{a_R} = \frac{2\alpha L}{\eta_L}; \qquad \sqrt{a_F} = \sqrt{a_R}\exp(2\alpha) \tag{7}$$

which yields

$$\eta_L = \frac{2\alpha L}{[\exp(2\alpha) - 1]\sqrt{a_R}} \tag{8}$$

Since the Fourier number Fo is defined by Eq. (6), we can estimate the distribution parameter α from the response time t_γ for the temperature ratio γ, the Fourier number $\text{Fo}_\gamma(\alpha)$ corresponding to t_γ, the thermal diffusivity a_R at the rear surface, and the thickness L of the FGM, through the following relation:

$$\frac{\text{Fo}_\gamma(\alpha)}{\text{Fo}_\gamma(0)}\left[\frac{2\alpha}{\exp(2\alpha) - 1}\right]^2 = \frac{a_R t_\gamma}{\text{Fo}_\gamma(0)\, L^2} \tag{9}$$

where $\text{Fo}_\gamma(0) = 0.4922$ for $\gamma = 2.5$, $\text{Fo}_\gamma(0) = 0.3034$ for $\gamma = 3.0$, and $\text{Fo}_\gamma(0) = 0.1907$ for $\gamma = 4.0$. Note that the right-hand side in Eq. (9) is an experimentally determinable quantity. Although the relation among α, $\text{Fo}_\gamma(\alpha)$, t_γ, a_R, and L is expressed by Eq. (9), a numerical calculation is required to obtain α because Eq. (9) is a transcendental equation in terms of α. Moreover, in obtaining α, we are also required to use the numerical result shown in Fig. 2. Then Eq. (9) itself is not so useful from the engineering point of view. An explicit expression is, therefore, required which is simple and is applicable for widely varying parameters.

For this purpose, we first obtain an approximate analytical expression for $[\text{Fo}_\gamma(\alpha)]/[\text{Fo}_\gamma(0)]$ with the perturbation method, which has been used to obtain the temperature response of the two-layered material [1] and that of the FGM [3]. If we let

$$\frac{\mathrm{Fo}_\gamma(\alpha)}{\mathrm{Fo}_\gamma(0)} = 1 + \varepsilon \tag{10}$$

we have a relation

$$\begin{aligned}
&2\mathrm{Fo}_\gamma(0)(1+\varepsilon) + \frac{1-\alpha\coth\alpha}{2\alpha^2} - 2\sum_{k=1}^{\infty}\frac{(-1)^k}{(k\pi)^2}\exp[-2(k\pi)^2\,\mathrm{Fo}_\gamma(0)(1+\varepsilon)] \\
&\quad\times\left(\frac{\sinh\alpha}{\alpha}\right)\exp[-2\alpha^2\mathrm{Fo}_\gamma(0)(1+\varepsilon)]\left[\frac{(k\pi)^2}{(k\pi)^2+\alpha^2}\right]^2 \\
&= \gamma\left\{\mathrm{Fo}_\gamma(0)(1+\varepsilon) + \frac{1-\alpha\coth\alpha}{2\alpha^2}\right. \\
&\quad - 2\sum_{k=1}^{\infty}\frac{(-1)^k}{(k\pi)^2}\exp[-(k\pi)^2\,\mathrm{Fo}_\gamma(0)(1+\varepsilon)] \\
&\quad\left.\times\left(\frac{\sinh\alpha}{\alpha}\right)\exp[-\alpha^2\mathrm{Fo}_\gamma(0)(1+\varepsilon)]\left[\frac{(k\pi)^2}{(k\pi)^2+\alpha^2}\right]^2\right\}
\end{aligned} \tag{11}$$

If we further use the mathematical formula of the Taylor expansion, we obtain the following relations.

0(1) Term:

$$\begin{aligned}
&2\mathrm{Fo}_\gamma(0) - \tfrac{1}{6} - 2\sum_{k=1}^{\infty}\frac{(-1)^k}{(k\pi)^2}\exp[-2(k\pi)^2\,\mathrm{Fo}_\gamma(0)] \\
&\quad = \gamma\left\{\mathrm{Fo}_\gamma(0) - \tfrac{1}{6} - 2\sum_{k=1}^{\infty}\frac{(-1)^k}{(k\pi)^2}\exp[-(k\pi)^2\,\mathrm{Fo}_\gamma(0)]\right\}
\end{aligned} \tag{12}$$

0(ε) Term:

$$\begin{aligned}
0 &= (\gamma-2)\,\mathrm{Fo}_\gamma(0)\,\varepsilon \\
&\quad + \left[(\gamma-2)\{\mathrm{Fo}_\gamma(0)\}^2 - \frac{(2\gamma-3)}{6}\,\mathrm{Fo}_\gamma(0) + \frac{7}{180}(\gamma-1)\right]\alpha^2 \\
&\quad + 2\gamma\sum_{k=1}^{\infty}(-1)^k\exp[-(k\pi)^2\,\mathrm{Fo}_\gamma(0)]\left\{\mathrm{Fo}_\gamma(0)\,\varepsilon + \frac{2\alpha^2}{(k\pi)^4}\right\} \\
&\quad - 2\sum_{k=1}^{\infty}(-1)^k\exp[-2(k\pi)^2\,\mathrm{Fo}_\gamma(0)] \\
&\quad \times\left\{2\mathrm{Fo}_\gamma(0)\,\varepsilon + \frac{\alpha^2}{(k\pi)^2}\,\mathrm{Fo}_\gamma(0)(1+\varepsilon) + \frac{2\alpha^2}{(k\pi)^4}\right\}
\end{aligned} \tag{13}$$

Since the contibutions of the first and second terms on the right-hand side of Eq. (13) are greater than the others, we can obtain an approximate expression for the perturbed term ε, and hence, $\mathrm{Fo}_\gamma(\alpha)$ is expressed as

$$\frac{\mathrm{Fo}_\gamma(\alpha)}{\mathrm{Fo}_\gamma(0)} \approx 1 - \left\{ \mathrm{Fo}_\gamma(0) - \frac{(2\gamma-3)}{6(\gamma-2)} + \frac{7(\gamma-1)}{180(\gamma-2)\,\mathrm{Fo}_\gamma(0)} \right\} \alpha^2 \tag{14}$$

This expression represents reasonably well the numerical results in Fig. 2 for the range of $-1 \leqslant \alpha \leqslant 1$ within 4% error for $\gamma = 4$. When $\gamma \leqslant 3$, Eq. (14) represents the numerical results within 1% error. Note that even at $\alpha = 1$ the ratio of A_F and A_R is 7.39, which is the same order of magnitude as that for any combinations of conventional solid materials.

Since $[\mathrm{Fo}_\gamma(\alpha)]/[\mathrm{Fo}_\gamma(0)]$ is expressed analytically, we next modify Eq. (14) so that we can obtain α explicitly. The mathematical formula for the Taylor expansion is again used, which yields the following relations:

$$\frac{\mathrm{Fo}_\gamma(\alpha)}{\mathrm{Fo}_\gamma(0)} \approx \exp\left(- \left\{ \mathrm{Fo}_\gamma(0) - \frac{(2\gamma-3)}{6(\gamma-2)} + \frac{7(\gamma-1)}{180(\gamma-2)\,\mathrm{Fo}_\gamma(0)} \right\} \alpha^2 \right) \tag{15}$$

$$\left[\frac{2\alpha}{\exp(2\alpha)-1} \right]^2 = \exp(-2\alpha) \left(\frac{\alpha}{\sinh \alpha} \right)^2 \approx \exp\left(-\frac{\alpha^2}{3} - 2\alpha \right) \tag{16}$$

Note that Eq. (15) represents reasonably well the numerical results shown in Fig. 2, for the range of $-1.5 \leqslant \alpha \leqslant 1.5$, within 2% error when the temperature ratio γ is 3 or less. When $\gamma = 4$, Eq. (15) represents the numerical results within 4% error for $-1 \leqslant \alpha \leqslant 1$. Substituting Eqs. (15) and (16) into Eq. (9), we have the following equation:

$$\left\{ \frac{1}{3} + \mathrm{Fo}_\gamma(0) - \frac{(2\gamma-3)}{6(\gamma-2)} + \frac{7(\gamma-1)}{180(\gamma-2)\,\mathrm{Fo}_\gamma(0)} \right\} \alpha^2 + 2\alpha + \ln\left(\frac{a_R t_\gamma}{\mathrm{Fo}_\gamma(0)\, L^2} \right) \approx 0 \tag{17}$$

from which we can easily obtain α. If we put

$$A = \frac{1}{3} + \mathrm{Fo}_\gamma(0) - \frac{(2\gamma-3)}{6(\gamma-2)} + \frac{7(\gamma-1)}{180(\gamma-2)\,\mathrm{Fo}_\gamma(0)}, \qquad C = \ln\left(\frac{a_R t_\gamma}{\mathrm{Fo}_\gamma(0)\, L^2} \right) \tag{18}$$

α is expressed as

$$\alpha = \left(-\frac{1}{A} \right) (1 - \sqrt{1 - AC}) \tag{19}$$

Another solution is excluded because its value is out of the range that we consider.

4. ESTIMATED PROFILE FOR THERMAL DIFFUSIVITY

In this section, we consider the estimation of the profile for thermal diffusivity with the relation expressed by Eq. (19). We consider a representative sample material of the FGM, the front surface of which is composed of Component I, with a thermal diffusivity of α_I, and the rear surface which is composed of Component II, with a thermal diffusivity of α_{II}. When the front surface of the FGM is subjected to stepwise heating, the distribution parameter α_{II} is obtained as

$$\alpha_{II} = \left(-\frac{1}{A}\right)\left[1 - \left\{1 - A \ln\left(\frac{a_{II} t_\gamma}{\text{Fo}_\gamma(0) L^2}\right)\right\}^{1/2}\right] \tag{20}$$

Since this α_{II} is determined with a_{II}, this α_{II} represents the profile for the thermal diffusivity near the rear surface.

We can also expect that the same response time is obtained even when we change the heating surface of the FGM. For this case, the distribution parameter α_I is obtained as

$$\alpha_I = \left(-\frac{1}{A}\right)\left[1 - \left\{1 - A \ln\left(\frac{a_I t_\gamma}{\text{Fo}_\gamma(0) L^2}\right)\right\}^{1/2}\right] \tag{21}$$

which represents the the profile for the thermal diffusivity near the front surface.

If we use Eqs. (20) and (21), we can express the profile for the thermal diffusivity as

$$\frac{a}{a_I} = \frac{a_{II}}{a_I} \exp(4\alpha_{II}\zeta) \qquad \text{for} \quad 0 \leqslant \zeta \leqslant \zeta_* \tag{22}$$

$$\frac{a}{a_I} = \exp[4\alpha_I(1-\zeta)] \qquad \text{for} \quad \zeta_* \leqslant \zeta \leqslant 1 \tag{23}$$

Here ζ_* is the cross point of these two curves and is expressed as

$$\zeta_* = \frac{1}{\alpha_I + \alpha_{II}}\left[\alpha_I + \frac{1}{4}\ln\left(\frac{a_I}{a_{II}}\right)\right] \tag{24}$$

We can also express the distance z from the rear surface of the FGM with ζ as

$$\left(\frac{-z}{L}\right) = \frac{\eta_L}{2\alpha_{II} L}\sqrt{a_{II}}\,[\exp(2\alpha_{II}\zeta) - 1] \qquad \text{for} \quad z_* \leqslant z \leqslant 0 \tag{25}$$

$$\left(\frac{-z}{L}\right)=\left(\frac{-z}{L}\right)-\frac{\eta_L}{2\alpha_I L}\{\sqrt{a_I}\exp[2\alpha_I(1-\zeta)]-\sqrt{a_*}\}\quad \text{for}\quad -L\leqslant z\leqslant z_* \tag{26}$$

where

$$\left(\frac{-z_*}{L}\right)=\frac{\eta_L}{2\alpha_{II}L}(\sqrt{a_*}-\sqrt{a_{II}}) \tag{27}$$

$$a_*=\exp\left(\frac{4\alpha_I\alpha_{II}}{\alpha_I+\alpha_{II}}+\frac{\alpha_I\ln a_{II}}{\alpha_I+\alpha_{II}}+\frac{\alpha_{II}\ln a_I}{\alpha_I+\alpha_{II}}\right) \tag{28}$$

$$\eta_L=2L\Big/\left[\left(\frac{\sqrt{a_I}-\sqrt{a_*}}{-\alpha_I}\right)+\left(\frac{\sqrt{a_*}-\sqrt{a_{II}}}{\alpha_{II}}\right)\right] \tag{29}$$

5. COMPARISONS

In our previous study [3], we investigated the temperature response for a multilayered material and/or an FGM which consists of Fe and TiO_2. We assumed that the rear surface is made of Fe, that the front surface is TiO_2, and that the mixture ratio of these components in the FGM is a linear function of the thickness. Thermophysical properties of the material which consists of Component I (TiO_2) with a mixture ratio X and Component II (Fe) with a mixture ratio $(1-X)$ were given by the following relations:

$$\rho=\rho_I X+\rho_{II}(1-X),\qquad \rho c=\rho_I c_I X+\rho_{II}c_{II}(1-X) \tag{30}$$

$$\frac{1}{\lambda}=\frac{X}{\lambda_I}+\frac{1-X}{\lambda_{II}},\qquad a=\frac{\lambda}{\rho c} \tag{31}$$

Discretization for the thermophysical properties was made for the multilayered material, keeping the same thickness for each layer.

The specification of our representative sample material was as follows: the thickness is $L=5\times 10^{-3}$ m, and the thermal diffusivities at the surfaces are $a_I=2.91\times 10^{-6}\ m^2\cdot s^{-1}$ and $a_{II}=2.27\times 10^{-5}\ m^2\cdot s^{-1}$. For a 10-layered material, the response time was obtained as $t_\gamma=1.542$ s for a temperature ratio of $\gamma=3.0$.

With this numerical result, let us estimate here the profile of the thermal diffusivity with the method mentioned in Section 4. The distribution parameters are determined as $\alpha_I=0.2500$ and $\alpha_{II}=-0.9375$. It is also determined as $\eta_L=1.973\ s^{1/2}$, $a_*=5.392\times 10^{-6}\ m^2\cdot s^{-1}$, $\zeta_*=0.3833$, and $(-z_*/L)=0.5139$.

Figure 3 shows the profile of the thermal diffusivity $\ln(a/a_R)$ as a function of ζ. Solid lines represent the estimated profile according to the present

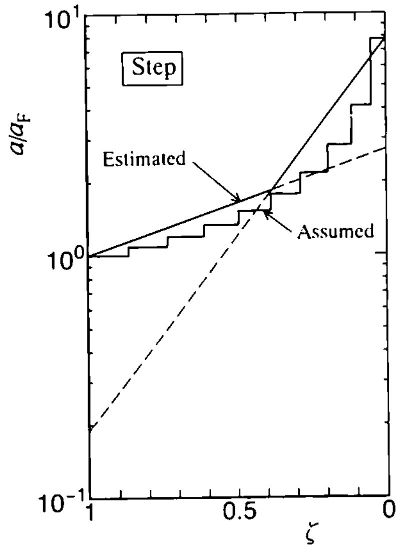

Fig. 3. Profile of the thermal diffusivity a in the FGM as a function of the normalized thermal diffusion time ζ. Solid lines are estimated by the present method; the stepwise profile is the given profile to obtain the response time, t.

method, while the profile of a was given by the stepwise function. We see that the estimation is reasonably good as far as the trend and the approximate magnitude are concerned. A plot of a as a function of $(-z/L)$ is given in Fig. 4.

6. CONCLUDING REMARKS

In the present study, the temperature response in a functionally gradient material (FGM) which is subjected to stepwise heating was investigated, to estimate and/or to evaluate the profile of thermal diffusivity from the temperature response at the rear surface of the FGM. Since the distribution parameter gives the profiles of thermophysical properties when an exact analytical solution exists for the temperature response in the FGM, emphasis has been placed on this parameter. When we consider the situation where the thermal diffusivity varies along the distance from the front surface of the FGM and the heat capacity is constant, an explicit expression determining the distribution parameter is obtained as a function of the thermophysical properties at the rear surface. This explicit expression

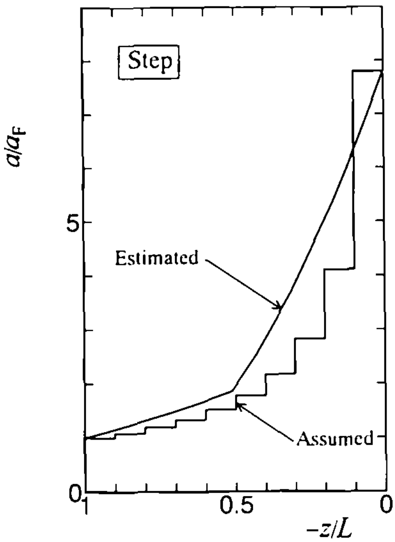

Fig. 4. Profile of the thermal diffusivity a in the FGM as a function of the normalized thickness ($-z/L$), obtained from the results in Fig. 3. The solid curve is estimated by the present method; the stepwise profile is the given profile to obtain the response time, t_r.

can represent reasonably well the dependence of the temperature response on the thermophysical properties within 5%. From an engineering point of view, because of its simplicity and fair degree of agreement, it is anticipated that this expression will be used widely. It is also considered that this identification of the explicit relation can provide useful insight into the estimation of the thermophysical properties of FGMs.

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NOMENCLATURE

a	Thermal diffusivity
c	Specific heat
Fo	Fourier number
f	Correction factor
L	Thickness of FGM
t	Time
V_S	Normalized temperature response
X	Mixture ratio
z	Distance

Greek Symbols

α	Distribution parameter
γ	Temperature ratio $[=V_S(2\mathrm{Fo})/V_S(\mathrm{Fo})]$
ε	Perturbed term
ζ	Normalized thermal diffusion time
η_L	Total thermal diffusion time
Λ	Heat-penetration coefficient $[=\lambda/\sqrt{a}]$
λ	Thermal conductivity
ρ	Density

Subscripts

F	Front surface
R	Rear surface
I	Component I
II	Component II
*	Cross point of estimated profiles

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